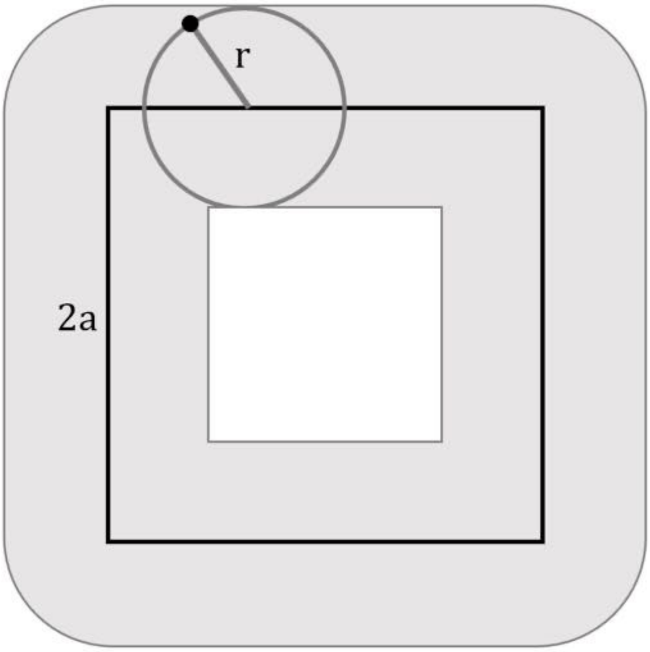
Questions

1. Spanning Areas

I have a pencil attached to a rope of length r, freely able to move about the pivot, which is in turn free to move along the perimeter of a square frame of side length $2a$. I use the pencil to shade in the area it spans. The diagram below illustrates the situation:



What is the area of the shaded shape? There are multiple cases based on the relationship between r and a.

2. Ordering Numbers

a) Without using a calculator, order the following numbers from smallest to largest:

$$\frac{2}{3} \quad \frac{3}{2} \quad \log_2(3) \quad \log_3(2) \quad 2^{\frac{1}{3}}$$

b) i) Find the stationary point of the function $y = x^{\frac{1}{x}}$. Note that $x^{\frac{1}{x}} = e^{\frac{1}{x}\ln(x)}$.

ii) You are told that this stationary point is a global maximum. Which number is greater, e^{π} or π^{e}?

Hints

1. Spanning Areas

- The key to this question is figuring out what the different cases are. Think about how large r has to be until certain regions begin to overlap.
- Drawing out your own diagram and separating the region into rectangles and segments of circles may help considerably.

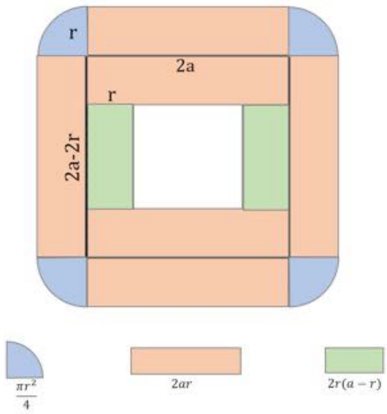
2. Ordering Numbers

- a) Use the fact that, for positive numbers, order is maintained when raising numbers to positive powers, i.e $x < y \iff x^a < y^a$
- Try to compare $\log_3(2)$ with $\frac{2}{3}$, and compare $2^{\frac{1}{3}}$ with $\frac{2}{3}$ and $\frac{3}{2}$.
- b) Which number is greater, $e^{\frac{1}{e}}$ or $\pi^{\frac{1}{\pi}}$? From this inequality, you can raise both numbers to the same power to achieve the desired answer.

Suggested answers

1. Spanning Areas

Working first with the diagram and situation given in the first question, we can split the shaded shape into regions of rectangles and circles:

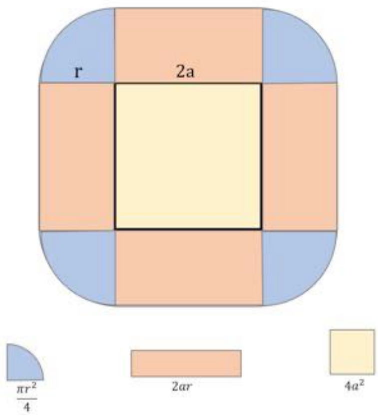


Summing these up, we get:

$$4 \cdot \frac{\pi r^2}{4} + 6 \cdot 2ar + 2 \cdot 2r(a-r)$$
$$= \pi r^2 + 4r(4a - r)$$

Now, for $r > a$, the circles spanned by the pencil and rope will begin to overlap with each other, and the entire interior square will be shaded in. Moreover, the regions we divided the shape up into in the previous case would not work, as when $r > a$, the green rectangles would have negative area.

Drawing a new diagram, we can split the shaded area into new rectangles and circle segments:



Summing up the new areas, we get:

$$4 \cdot \frac{\pi r^2}{4} + 4 \cdot 2ar + 4a^2$$
$$= \pi r^2 + 4a(a + 2r)$$

So, for $r \leq a$, the area is $\pi r^2 + 4r(4a - r)$, and when $r > a$, the area is $\pi r^2 + 4a(a + 2r)$.

2. Ordering Numbers

a) Certain orderings can be immediately deduced. For example, since the base is greater than the argument, $\log_3(2) < 1$. Similarly, we know that $\frac{2}{3} < 1 < \frac{3}{2}$ and $1 < \log_2(3) = \frac{1}{\log_3(2)}$. We also know that $1 < 2$, so $1^{\frac{1}{3}} = 1 < 2^{\frac{1}{3}}$.
We will first start by comparing $\log_3(2)$ and $\frac{2}{3}$. We can exponentiate with base 3 to get:

$$3^{\log_3(2)=2}$$
$$3^{\frac{2}{3}}$$

Cubing both numbers,

$$\begin{aligned} &2^3 = 8 \\ &(3^{\frac{2}{3}})^3 = 3^{3\cdot\frac{2}{3}} = 3^2 = 9 \end{aligned}$$

We know that $8 < 9$, so reversing these operations, we deduce that $\log_3(2) < \frac{2}{3}$. Since $\frac{2}{3}$ and $\log_3(2)$ are reciprocals of $\frac{3}{2}$ and $\log_2(3)$ respectively, we also deduce that $\frac{3}{2} < \log_2(3)$. Now we can compare $2^{\frac{1}{3}}$ to $\frac{2}{3}$ and $\frac{3}{2}$ by cubing:

$$\begin{aligned} &(2^{\frac{1}{3}})^3 = 2^{3\cdot\frac{1}{3}} = 2 \\ &(\frac{2}{3})^3 = \frac{8}{27} < 2 \\ &(\frac{3}{2})^3 = \frac{27}{8} = 3\frac{3}{8} > 2 \end{aligned}$$

From this, we deduce that $\frac{2}{3} < 2^{\frac{1}{3}} < \frac{3}{2}$. Thus, we have:

$$\log_3(2) < \frac{2}{3} < 2^{\frac{1}{3}} < \frac{3}{2} < \log_2(3)$$

In fact, in order, these numbers are approximately:

$$0.631 < 0.66\overline{6} < 1.260 < 1.5 < 1.585$$

b) Using the equality given in the question, we can apply the chain and product rule to find the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{du}(e^u) \cdot \frac{d}{dx}(\frac{1}{x}\ln(x)) \qquad\qquad u = \frac{1}{x}\ln(x) \\ &= e^{\frac{1}{x}\ln(x)} \cdot (\frac{-1}{x^2}\ln(x) + \frac{1}{x}\cdot\frac{1}{x}) \\ &= \frac{x^{\frac{1}{x}}}{x^2}(\ln(x) - 1) \end{aligned}$$

A stationary point will have $\frac{dy}{dx} = 0$:

$$
\begin{aligned}
\frac{x^{\frac{1}{x}}}{x^2}(\ln(x) - 1) &= 0 \\
\Rightarrow \ln(x) - 1 &= 0 \\
\Rightarrow \ln(x) &= 1 \\
x &= e
\end{aligned}
$$

So $(e, e^{\frac{1}{e}})$ is a stationary point.

Since we are told this is a global maximum, we have:

$$e^{\frac{1}{e}} > x^{\frac{1}{x}}$$

for all $x > 0$. In particular,

$$
\begin{aligned}
e^{\frac{1}{e}} &> \pi^{\frac{1}{\pi}} \\
(e^{\frac{1}{e}})^{e\pi} &> (\pi^{\frac{1}{\pi}})^{e\pi} \\
e^{\frac{e\pi}{e}} &> \pi^{\frac{e\pi}{\pi}} \\
e^{\pi} &> \pi^{e}
\end{aligned}
$$

So e^π is **greater than** π^e.

In fact, $e^\pi \approx 23.141$ and $\pi^e \approx 22.459$.