Questions

1. a) In a running hourglass, what forces keep grains of sand at the edge from moving?

   b) With reference to Newton's laws, explain how the sand adds to the mass reading of the hourglass.

   c) An hourglass is placed on a high-resolution balance. Initially, we weigh the mass of the hourglass when all the sand is located in the lower reservoir. We then place it upside down. How would the reading of the mass balance change with time? Sketch and explain a graph of mass reading against time for the hourglass, and state any assumptions made.

2. a) I have N six-sided dice, I roll all of them at once, and remove any dice which show a six. I then roll the remaining dice, and again remove all dice showing a six. I roll the dice a total number of t times. Derive an expression for the expected number of dice after t rolls.

   b) Find, on average, the number of rolls until I have no dice left.

   c) Which physical phenomena could be modelled by the dice in the way outlined in part a)?

Hints

Question 1

c) i) Break the problem up into three phases.

ii) Think about what happens as sand leaves the top half of the hourglass, before it hits the bottom.

iii) Consider a collision between a grain of sand and the base of the hourglass.

Question 2

a) i) Try and form a first order differential equation.

ii) What do we know about the first derivative of the number of dice with respect to the number of 'turns'?

b) Is the function you have found continuous? Are the variables we are trying to model continuous or discrete?

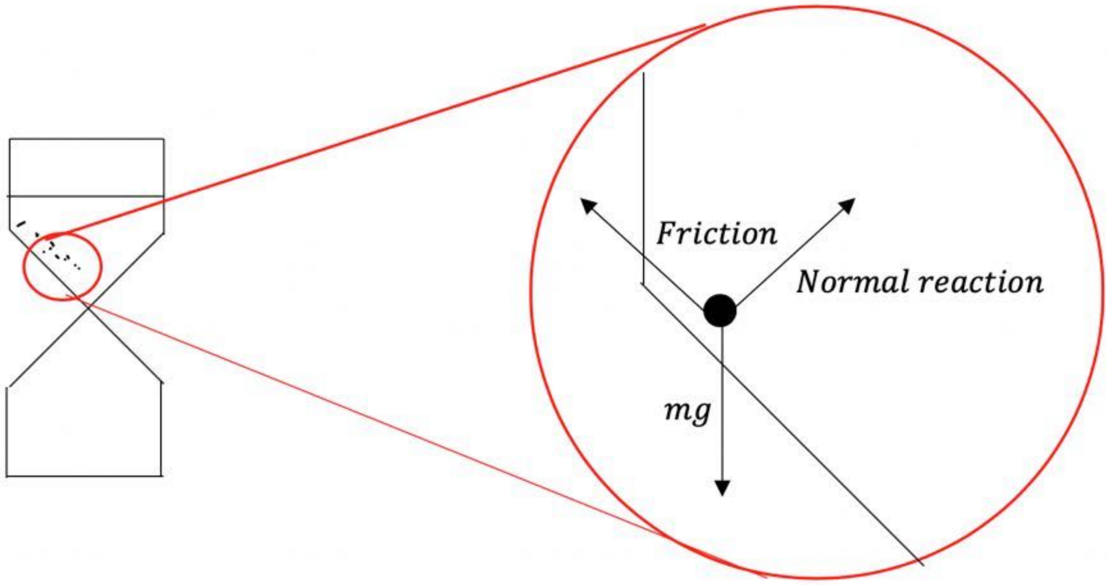
Suggested answers

Question 1

a)

- The stationary grains of sand are held in place by a combination of reaction forces and frictional forces from the sides of the hourglass. These two forces combined are balanced with the weight force, since the sand grain is stationary and in equilibrium.

- Consider the force on a single grain of sand in the diagram below:



- Reaction forces must act normally to the plane, and, in hourglasses, the surface is often angled to the horizontal curves to make sand flow easily.
    - Hence, for a grain of sand to be in equilibrium, the grain must be affected by both the normal reaction force and friction, as weight acts vertically downwards.
    - Any grains of sand above this grain, not in contact with the surfaces of the hourglass, will be held in place by the normal reaction force and friction from other grains of sand. These sand grain – sand grain interactions are unable to hold sand grains up themselves, as a consequence of Newton's third law.

b)

- The balance measures the mass by using the reaction force required to keep the object in equilibrium. A good way to visualise this concept is to consider the balance as a surface held up by springs; the reaction force required to keep the mass in equilibrium could be measured by calculating the compression of the spring.
    - The grains of sand at the bottom of the hourglass are 'held up' by the reaction force from the base of the hourglass, which is equal to the weight of the sand.

- By Newton's third law – every action has an equal and opposite reaction - we know that the grain of sand exerts an equal and opposite force on the base of the hourglass.
    - Hence, the balance must exert extra force to compensate for the weight of the sand, which adds to the mass reading of the hourglass.

c)

- Let's first begin by saying that the weight of the measured mass of the hourglass, before it is upturned, is m_0.

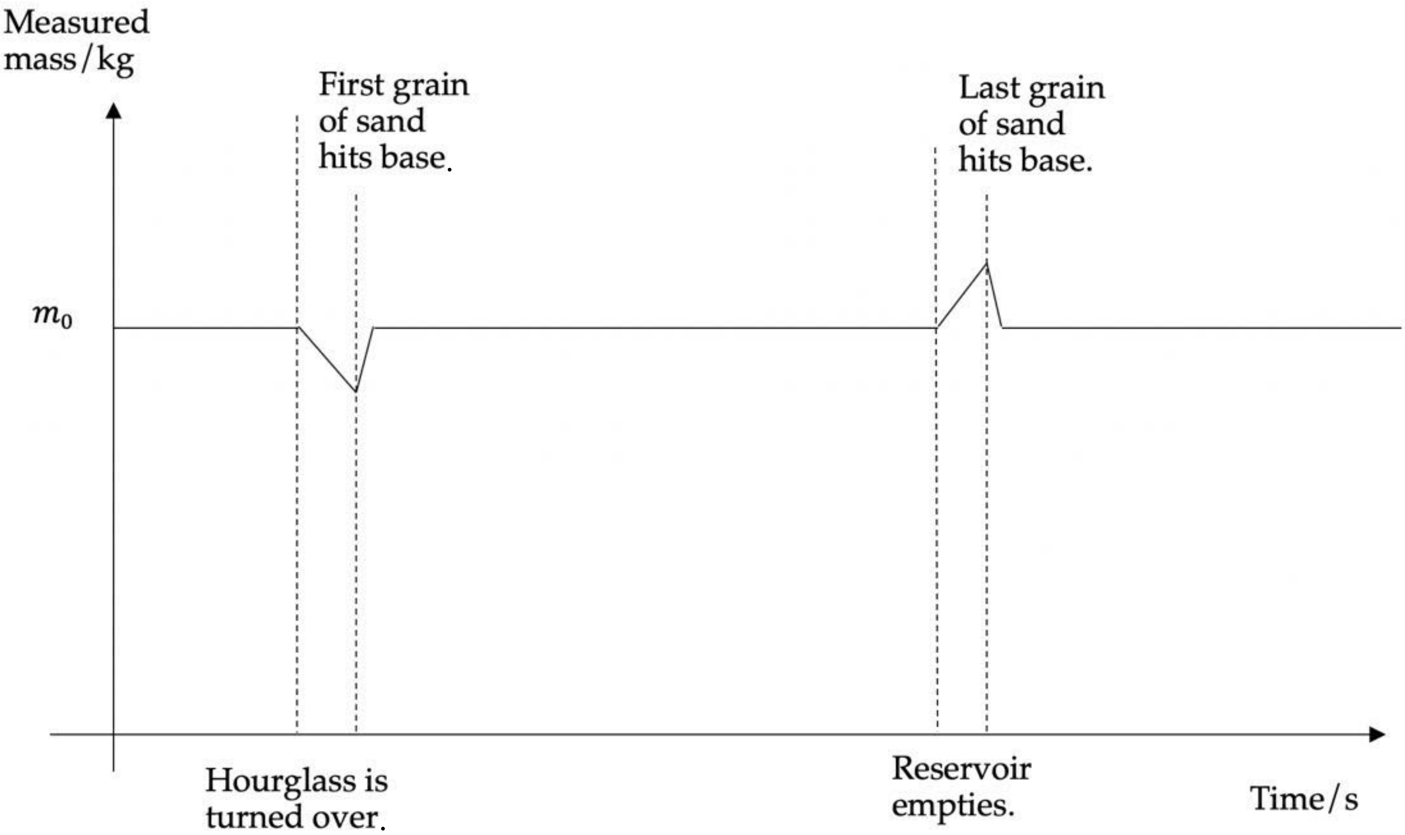
- Now consider the instant when the mass is upturned and on the balance:
    - Some of the sand grains begin to accelerate under gravity.
        - This logically means that the reaction force provided by the side of the hourglass is smaller than the previous reaction force (when the hourglass wasn't upturned).
        - Hence the mass reading will be $< m_0$.
    - As more and more sand grains begin to move under free fall but have not yet hit the base of the hourglass, the measured mass of the hourglass must become increasingly smaller, by the same logic.
    - Another reasonable assumption is that, as the sand begins falling linearly, the rate at which the sand comes into freefall is constant.
        - In other words, there is a constant number of grains per second which begin to freefall.

- It now makes sense to consider what happens when sand grains collide with the base of the hourglass after being in freefall:
    - We need to assume that, when the sand hits the base of the hourglass, it remains on the base.
        - This is a rather poor assumption, but we are not looking to do any calculations.
    - As the sand lands on the base, the base must exert a force on the sand.
        - The reaction force provided by the base of the hourglass must be significantly greater than the weight of the sand, because the sand has a downwards velocity and needs to accelerate upwards until the sand is at rest. This would require a force much greater than the weight of the sand grain, as the resultant force needs to act upwards in order to cause upwards acceleration.
    - It is fair to assume that this large force must act over a short period of time, far smaller than the time taken by a sand grain to fall to the bottom of the hourglass.
        - This extra force would cause the mass measurement to be larger than it was previously - by Newton's third law.

- Now consider the sustained time period for which the timer is running (i.e. the interval in which sand is simultaneously hitting the base of the hourglass, and more sand is begging to freefall):
    - The sand falls at a constant rate, so, throughout this period:
        - Sand must hit the base at a constant rate.
        - We assume that each grain of sand undergoes the increased reaction force for the same period of time.
        - The number of grains in freefall at any given time throughout this time period is the same.
    - These assumptions lead to a sustained period of 'equilibrium' between these two effects, resulting in a constant mass reading.

- Now consider the final period of time, where no more sand is beginning to freefall, but sand is still hitting the base of the hourglass:
    - There will be a brief time period, towards the end, when the number of grains of sand in free fall decreases.
        - This will occur when the reservoir of sand within the top half of the hourglass is empty.
    - This increases the mass reading of the hourglass.
    - It will increase the mass reading until no more grains are in freefall.
        - The mass value will reach a brief maximum just when this happens.
    - The mass will then decrease to m_0, until the base of the hourglass has stopped exerting a large reaction force on the grains of sand. This reaction force reduces, such that the sand grains are in equilibrium when they are at rest.

- All of these points lead to a suggested sketch below:



Question 2

a)

- The obvious fact from the first part is that, on average, five sixths of the dice will remain from the last turn.

  This can be denoted as $\frac{dn}{dt} = -\frac{1}{6}n$.
  - Where n is the remaining number of dice remaining after t turns.

- Rearranging this equation gives

$$\frac{dn}{dt} + \frac{1}{6}n = 0$$

You can use the fact that five sixths of the dice will remain from the last turn throughout, when looking for the average number of turns (the expected value).

- This equation can be solved using an integrating factor as it is in the form:

$$\frac{dn}{dt} + P(t)n = Q(t).$$

If you have studied this kind of differential equation before, you know that we can find the integrating factor by using equation $u(x) = e^{\int P(x)dx}$:

$$u(t) = e^{\int \frac{1}{6}\,dt}$$

$$= e^{\frac{1}{6}t}$$

Multiplying the differential equation by the integrating factor:

$$\frac{dn}{dt}e^{\frac{1}{6}t} + \frac{1}{6}ne^{\frac{1}{6}t} = 0$$

$$\Longrightarrow \frac{d}{dt}\left(ne^{\frac{1}{6}t}\right) = 0$$

Integrating this expression:

$$ne^{\frac{1}{6}t} = C$$

$$\Longrightarrow n = Ce^{-\frac{1}{6}t}$$

- We know that the integration constant, C, must be equal to N.
  - Since, after zero turns, we haven't lost any dice, so $n = N$.
  - Therefore, the expression we are looking for is $n = Ne^{-\frac{1}{6}t}$.

b)

- Notice that the function generated is, in fact, continuous, even though the variable being considered, n, is discrete, because it doesn't make sense to have a fraction of a die left.

- The function will never actually reach zero.
  - Therefore, we cannot solve the equation for $n = 0$.

- As a result of the above considerations, we should actually solve for $n = \frac{1}{2}$.
  - This is because, at this point, it is more likely than not that there are 0 dice.

$$0.5 = Ne^{-\frac{1}{6}t}$$

$$\frac{1}{2N} = e^{-\frac{1}{6}t}$$

$$\ln\frac{1}{2N} = -\frac{1}{6}t$$

Using $\ln\frac{1}{x} = -\ln x$.

$$6\ln 2N = t$$

If this equation were used numerically, and gave a non-integer number, you should round up.

c)

- The equation derived is very similar to the nuclear decay equation $N = N_0e^{-kt}$.
  - Therefore, the dice could be used to model nuclei with decay constant $k = \frac{1}{6}$, where the number of turns represents each passing second.

- Alternatively, any other physical situation in which there is an exponential decay could also be modelled by the dice.
  - However, nuclear decay is a very good comparison to the dice experiment.